

A rational approach to resonance saturation in large- N_c QCD

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ABSTRACT: We point out that resonance saturation in QCD can be understood in the large- N_c limit from the mathematical theory of Pade Approximants to meromorphic functions. These approximants are rational functions which encompass any saturation with a finite number of resonances as a particular example, explaining several results which have appeared in the literature. We review the main properties of Pade Approximants with the help of a toy model for the $\langle VV - AA \rangle$ two-point correlator, paying particular attention to the relationship among the Chiral Expansion, the Operator Product Expansion and the resonance spectrum. In passing, we also comment on an old proposal made by Migdal in 1977 which has recently attracted much attention in the context of AdS/QCD models. Finally, we apply the simplest Pade Approximant to the $\langle VV - AA \rangle$ correlator in the real case of QCD. The general conclusion is that a rational approximant may reliably describe a Green's function in the Euclidean, but the same is not true in the Minkowski regime due to the appearance of unphysical poles and/or residues.

KEYWORDS: $1/N$ Expansion, QCD, Chiral Lagrangians.

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1. Introduction

The strong Chiral Lagrangian is a systematic organization of the physics in powers of momenta and quark masses, but requires knowledge of the low-energy constants (LEC) to make reliable phenomenological predictions. As with any other effective field theory, these LECs play the role of coupling constants and contain the information which comes from the integration of the heavy degrees of freedom not explicitly present in the Chiral Lagrangian (e.g. meson resonances).

At $\mathcal{O}(p^4)$ there are 10 of these constants [1]. Although at this order there is enough independent information to extract the values for these constants from experiment, this will hardly ever be possible at the next order, $\mathcal{O}(p^6)$, because the number of constants becomes more than a hundred [2]. In the electroweak sector the proliferation of constants appears already at $\mathcal{O}(p^4)$ [3]. Although in principle these low-energy constants may be computed on the lattice, in practice this has only been accomplished in a few cases for the strong Chiral Lagrangian at $\mathcal{O}(p^4)$, and only recently [4].

The large N_c expansion [5] stands out as a very promising analytic approach capable of dealing with the complexities of nonperturbative QCD while, at the same time, offering a relatively simple and manageable description of the physics. For instance, mesons are $q\bar{q}$ states with no width, the OZI rule is exact and there is even a proof of spontaneous chiral symmetry breaking [6]. Furthermore, its interest has recently received a renewed boost indirectly through the connection of some highly supersymmetric gauge theories to

gravity [7], although the real relevance of this connection for QCD still remains to be seen. However, in spite of all this, the fact that no solution to large- N_c QCD has been found keeps posing a serious limitation to doing phenomenology. For instance, in order to reproduce the parton model logarithm which is present in QCD Green's functions in perturbation theory, an infinity of resonances is necessary whose masses and decay constants are in principle unknown.

On the other hand, QCD Green's functions seem to be approximately saturated by just a few resonances; a property which has a long-standing phenomenological support going all the way back to vector meson dominance ideas [8], although it has never been properly understood. In a modern incarnation, this fact translated into the very successful observation [9] that the strong LECs at $\mathcal{O}(p^4)$ seem to be well saturated by the lowest meson in the relevant channel,¹ after certain constraints are imposed on some amplitudes at high-energy in order to match the expected behavior in QCD [11, 12]. It was then realized that all these successful results could be encompassed at once as an approximation to large- N_c QCD consisting in keeping only a *finite* (as opposed to the original infinite) set of resonances in Green's functions. This approximation to large- N_c QCD has been termed Minimal Hadronic Approximation (MHA) [13–15] because it implements the minimal constraints which are necessary to secure the leading non-trivial behavior at large energy of certain Green's functions through the marriage of the old resonance saturation and the large- N_c approximation of QCD. In recent years, a large amount of work has been dedicated to studying the consequences of these ideas [16].

However, the high-energy matching with a finite set of resonances, first suggested in [11], makes it clear that the treatment is not amenable to the methods of a conventional effective field theory. An effective field theory is an approximation for energies smaller than a heavy particle's mass and, therefore, cannot deal with momentum expansions at infinity as in the case of the Operator product Expansion (OPE). In other words, the fact that the set of resonances in each channel is really infinite precludes the naive expansion at large momentum because there is always a mass in the spectrum which is even larger. The sum over an infinite set of resonances and the expansion for large momentum are operations which do not commute [17]. In those Green's functions containing a contribution from the parton model logarithm, this is made self-evident since a naive expansion at large momentum can only produce powers and not a logarithm, which is why large- N_c QCD requires an infinity of resonances in the first place.

The problem can be delayed one power of α_s if one requires the use of the resonance Lagrangian [9] to be limited only to Green's functions which are order parameters of spontaneous chiral symmetry breaking. These order parameters vanish to all orders in α_s in the chiral limit² and, therefore, avoid the presence of the parton model logarithm which, otherwise, would preclude from the outset any matching to a finite number of resonances. However, the concept of a Lagrangian whose validity is restricted only to a certain class of Green's functions has never been totally clear; and even if the resonance Lagrangian

¹This is less clear in the scalar channel, however. See ref. [10]

²E.g., the two-point correlator $\langle VV - AA \rangle$.

is restricted by definition to order parameters, the problem surfaces again in the presence of logarithmic corrections from nontrivial anomalous dimensions, which make the exact matching at infinite momentum impossible.

In a slightly different context, a somewhat similar observation was also made in ref. [18]. In this paper it was observed that it is impossible to satisfy the large momentum fall-off expected in large- N_c QCD for the form factors which can be defined through a three-point Green's function, if the sum over resonances in the Green's function is restricted to a finite set. Interestingly, this again pointed to an incompatibility of the QCD short-distance behavior with an approximation to large N_c which only kept a finite number of resonances.

A further piece of interesting evidence results from the comparison between the analysis in refs. [19] and [11]. After imposing some good high-energy behavior in several Green's functions and form factors including, in particular, the axial form factor governing the decay $\pi \rightarrow \gamma e \bar{\nu}$, ref. [11] obtains, keeping only one vector state V and one axial-vector state A , that their two masses must be constrained by the relation $M_A = \sqrt{2}M_V$. The work in ref. [19], on the contrary, does not use the axial form factor and obtains, after performing a very good fit within the same set of approximations, the precise values $M_V = 775.9 \pm 0.5$ MeV and $M_A = 938.7 \pm 1.4$ MeV. These values for the masses, although close, are not entirely compatible with the previous relation. In other words, the short-distance constraint from the axial form factor is not fully compatible with the short-distance constraints used in [19] if restricted to only one vector and one axial-vector states.³

In this paper we would like to suggest that all the above properties can be understood if the approximation to large N_c QCD with a finite number of resonances is reinterpreted within the mathematical Theory of Pade Approximants (PA) to meromorphic functions [20, 21]. QCD Green's functions in the large N_c and chiral limits have an analytic structure in the complex momentum plane which consists of an infinity of isolated poles but no cut, i.e. they become meromorphic functions [15]. As such, they have a well-defined series expansion in powers of momentum around the origin with a finite radius of convergence given by the first resonance mass.⁴ This is all that is needed to construct a Pade Approximant.⁵ A theorem by Pommerenke [22] assures then convergence of any near diagonal PA to the true function for any finite momentum, over the whole complex plane, except perhaps in a zero-area set. The poles of the original Green's function (i.e. the resonance masses) belong to this zero-area set because not even the original function is defined there, but there are also *extra* poles. These extra poles are called “defects” in the mathematical literature [20, 21]. When the Green's function being approximated is of the Stieltjes type,⁶ the poles of the PA are always real and located on the Minkowski region $\text{Re}(q^2) = \text{Re}(-Q^2) > 0$, approaching the physical poles as the order of the PA is increased [23]. However, this takes place in a

³Adding one further state does not change the conclusion [19].

⁴The pion pole can always be eliminated multiplying by enough powers of momentum. We are assuming here the existence of a nonvanishing gap in large- N_c QCD.

⁵Except for the appendix, in this paper we will construct our rational approximants using as input the expansion around the origin.

⁶Roughly this means that the associated spectral function is positive definite, like in the case of the two-point correlator $\langle VV \rangle$. See ref. [20] (chapter 5) for a more precise definition.

hierarchical way and, while the poles in the PA which are closest to the origin are also very close to the physical masses, the agreement quickly deteriorates and one may find that the last poles are several times bigger than their physical counterparts.⁷ The same is true of the residues. In section 3, we will see with the help of a model that the same properties are met in a meromorphic function whose spectral function is not positive definite, except that some of the poles in the PA may even be complex.

This means that Minkowskian properties, such as masses and decay constants, cannot be reliably determined from a PA except, perhaps, from the first poles which are closest to the origin. If not all the residues and/or masses are physical, then there is no reason why they should be the same in the form factor governing $\pi \rightarrow \gamma e\bar{\nu}$ and in the Green's function $\langle VV - AA \rangle$, explaining the different results found in [11] and [19] we alluded to above. Furthermore, the form factors of all but the lightest mesons, defined through the residues of the corresponding 3-point Green's functions, will not be reliably determined from a PA to that Green's function, again in agreement with the findings in [18].

The situation in the Euclidean is different. In general, PAs cannot be expanded at infinite momentum to generate an OPE type expansion for the true function. Nevertheless, Pommerenke's theorem assures a good approximation at any finite momentum, no matter how large. Of course, the order of the PA will have to increase, the larger the momentum region one wishes to approximate. For instance, in ref. [23] it was shown with the help of a simple model how, even in the case of the $\langle VV \rangle$ correlator which contains a $\log Q^2$ at large values of $Q^2 > 0$, the PAs are capable of approximating the true function at any arbitrarily large (but finite) value of $Q^2 > 0$, *without* the need for a perturbative continuum. In section 3 we will show, again with the help of a model, how this is also true in the more general case of a non-positive definite spectral function such as $\langle VV - AA \rangle$. This means that PAs are a reliable way to approximate the original Green's function in the Euclidean but not in the Minkowskian regime.

In 1977, A.A. Migdal [24] suggested PAs as a method to extract the spectrum of large- N_c QCD from the leading term in the OPE of the $\langle VV \rangle$ correlator, i.e. from the parton model logarithm. However, nowadays this proposal should be considered unsatisfactory for a number of reasons [25], the most simple of them being that different spectra may lead to the same parton model logarithm [26]. In fact, the full OPE series is expected to be only an asymptotic expansion at $Q^2 = \infty$ (i.e. with zero radius of convergence), and PAs constructed from this type of expansions cannot in general reproduce the position of the physical poles.⁸ For instance, we show this explicitly with the help of a model for $\langle VV - AA \rangle$ in the appendix. Migdal's approach has been recently adopted (in disguise) in some models exploiting the so-called AdS/QCD correspondence [27] and, consequently, the same criticism also applies to them.

In ref. [28] a model for the $\langle VV - AA \rangle$ two-point correlator with a spectrum consisting of an infinity of resonances was suggested as a theoretical laboratory for studying the relationship between the spectrum and the coefficients of the OPE. In this paper several

⁷See the table in ref. [23]

⁸Ref. [20], section 5.5.

conventional methods usually employed in the literature were tested against the exact result from the model. These included: Finite Energy Sum Rules as in ref. [32], pinched weights as in ref. [33], Laplace transforms as in ref. [34] and, finally, also resonance saturation as in the MHA method. The bottom line was that no method was able to produce very accurate predictions for the OPE coefficients. In all the methods but the last one, the reason for this lack of accuracy was basically due to the fact that the OPE requires an integral over the whole spectrum, whereas the integral is actually cut off at an upper limit (in the real case, the upper limit is m_τ). This is why even if one uses the real spectrum the result may be inaccurate [29]. In the case of the MHA the reason was, as we will comment upon below, that the poles were not allowed to be complex.

In section 3 we will revisit this $\langle VV - AA \rangle$ model, now from the point of view of PAs. The model reproduces the power behavior of QCD at large $Q^2 > 0$ except that the model is simple enough not to have any $\log Q^2$ and, therefore, it cannot reproduce the nonvanishing anomalous dimensions which exist in QCD. We do not think this is a major drawback, however, because in QCD these logarithms are always screened by at least one power of α_s and, hence, in an approximate sense, it may be licit to ignore them. In the model such an approximation becomes exact.⁹ Will the PAs be able to reproduce the large Q^2 expansion of the $\langle VV - AA \rangle$ model? We will see that the answer is affirmative. Therefore, the reason why the MHA method was not able to predict accurately the OPE coefficients in ref. [28] is because the lowest PA has complex poles which were not allowed in [28]. When these complex poles are considered, the accuracy achieved is better and, most importantly, improves for a higher PA. Since the model allows the construction of PAs of a very high order, we have checked this convergence up to the Pade P_{52}^{50} , which is able to reproduce the first non vanishing coefficient of the OPE in the model with an accuracy of 52 decimal figures. Together with other numerical examples which will be discussed in section 3, we take this as a clear evidence of the convergence of the method. This renders some confidence that PAs may also do a good job in the real case of QCD.

The rest of the paper is organized as follows. In section 2 we review some generalities of rational approximants, in section 3 we describe the $\langle VV - AA \rangle$ model and apply different rational approximants to learn about the possible advantages and disadvantages of them. In section 4 we apply the simplest PA to the case of the real $\langle VV - AA \rangle$ two-point function in QCD. Finally, we close with some conclusions.

2. Rational approximations: generalities

Let a function $f(z)$ have an expansion around the origin of the complex plane of the form

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad z \rightarrow 0. \quad (2.1)$$

One defines a Pade Approximant (PA) to $f(z)$, denoted by $P_N^M(z)$, as a ratio of two polynomials $Q_M(z), R_N(z)$,¹⁰ of order M and N (respectively) in the variable z , with a

⁹For a model with a $\log Q^2$, the reader may consult ref. [23].

¹⁰Without loss of generality we define, as it is usually done, $R_N(0) = 1$.

contact of order $M + N$ with the expansion of $f(z)$ around $z = 0$. This means that, when expanding $P_N^M(z)$ around $z = 0$, one reproduces exactly the first $M + N$ coefficients of the expansion for $f(z)$ in eq. (2.1):

$$P_N^M(z) = \frac{Q_M(z)}{R_N(z)} \approx f_0 + f_1 z + f_2 z^2 + \dots + f_{M+N} z^{M+N} + \mathcal{O}(z^{N+M+1}) . \quad (2.2)$$

At finite z , the rational function $P_N^M(z)$ constitutes a resummation of the series (2.1). Of special interest for us will be the case when $N = M + k$, for a fixed k , because then the function behaves like $1/z^k$ at $z = \infty$. The corresponding PAs $P_{M+k}^M(z)$ belong to what is called the near-diagonal sequence for $k \neq 0$, with the case $k = 0$ being the diagonal sequence.

The convergence properties of the PAs to a given function are much more difficult than those of normal power series and this is an active field of research in Applied Mathematics. In particular, those which concern meromorphic functions¹¹ are rather well-known and will be of particular interest for this work. The main result which we will use is Pommerenke's Theorem [22] which asserts that the sequence of (near) diagonal PA's to a meromorphic function is convergent everywhere in any compact set of the complex plane except, perhaps, in a set of zero area.¹² This set obviously includes the set of poles where the original function $f(z)$ is clearly ill-defined but there may be some other extraneous poles as well. For a given compact region in the complex plane, the previous theorem of convergence requires that, either these extraneous poles move very far away from the region as the order of the Pade increases, or they pair up with a close-by zero becoming what is called a *defect* in the mathematical jargon.¹³ These are to be considered artifacts of the approximation. Near the location of these extraneous poles the PA approximation clearly breaks down but, away from these poles, the approximation is safe.

In the physical case the original function $f(z)$ will be a Green's function $G(Q^2)$ of the momentum variable Q^2 . In QCD in the large N_c limit this Green's function is meromorphic with all its poles located on the negative real axis in the complex Q^2 plane. These poles are identified with the meson masses. On the other hand, the region to be approximated by the PAs will be that of euclidean values for the momentum, i.e. $Q^2 > 0$. The expansion of $G(Q^2)$ for Q^2 large and positive coincides with the Operator Product Expansion.

In general a meromorphic function does not obey any positivity constraints and, as we will see, this has as a consequence that some of the poles and residues of the PAs may become *complex*.¹⁴ This clearly precludes any possibility that these poles and residues may have anything to do with the physical meson masses and decay constants. However, and this is very important to realize, this does *not* spoil the validity of the rational approximation provided the poles, complex or not, are not in the region of Q^2 one is interested in. It is to be considered rather as the price to pay for using a rational function, which has only a

¹¹A function is said to be meromorphic when its singularities are only isolated poles.

¹²To be more precise, in a set of zero Hausdorff measure. See ref. [20], section 6.6.

¹³See [21], chapter 14, corollary 14.3.

¹⁴A special case which does obey positivity constraints is when the function is Stieltjes. In this case the poles and residues of the PAs are purely real and with the same sign as those of the original function [23].

finite number of poles, as an approximation to a meromorphic function with an infinite set of poles.

When the position of the poles in the original Green's function is known, at least for the lowest lying states, it is interesting to devise a rational approximation which has this information already built in. The corresponding approximants are called Partial Pade Approximants (PPAs) in the mathematical literature [30] and are given by a rational function $\mathbb{P}_{N,K}^M(Q^2)$:

$$\mathbb{P}_{N,K}^M(Q^2) = \frac{Q_M(Q^2)}{R_N(Q^2) T_K(Q^2)}, \quad (2.3)$$

where $Q_M(Q^2)$, $R_N(Q^2)$ and $T_K(Q^2)$ are polynomials of order M , N and K (respectively) in the variable Q^2 . The polynomial $T_K(Q^2)$ is defined by having K zeros precisely at the location of the lowest lying poles of the original Green's function¹⁵ i.e.

$$T_K(Q^2) = (Q^2 + M_1^2) (Q^2 + M_2^2) \dots (Q^2 + M_K^2). \quad (2.4)$$

As before the polynomial $R_N(Q^2)$ is chosen so that $R_N(0) = 1$ and, together with $Q_M(Q^2)$, they are defined so that the ratio $\mathbb{P}_{N,K}^M(Q^2)$ matches exactly the first $M + N$ terms in the expansion of the original function around $Q^2 = 0$, i.e. :

$$\mathbb{P}_{N,K}^M(Q^2) \approx f_0 + f_1 Q^2 + f_2 Q^4 + \dots + f_{M+N} Q^{2M+2N} + \mathcal{O}(Q^{2N+2M+2}). \quad (2.5)$$

At infinity, the PPA in eq. (2.3) obviously falls off like $1/Q^{2N+2K-2M}$. Exactly as it happens in the case of PAs, also the PPAs will have complex poles for a general meromorphic function, which prevents it from any interpretation in terms of meson states.

Finally, another rational approximant defined in mathematics is the so-called Pade Type Approximant (PTA) [30] $\mathbb{T}_N^M(Q^2)$:

$$\mathbb{T}_N^M(Q^2) = \frac{Q_M(Q^2)}{T_N(Q^2)}, \quad (2.6)$$

where $T_N(Q^2)$ is also given by the polynomial (2.4), now with N preassigned zeros at the corresponding position of the poles of the original Green's function, $G(Q^2)$. The polynomial $Q_M(Q^2)$ is defined so that the expansion of the PTA around $Q^2 = 0$ agrees with that of the original function up to and including terms of order $M + 1$, i.e.

$$\mathbb{T}_N^M(Q^2) \approx f_0 + f_1 Q^2 + f_2 Q^4 + \dots + f_M Q^{2M} + \mathcal{O}(Q^{2M+2}). \quad (2.7)$$

At large values of Q^2 , one has that $\mathbb{T}_N^M(Q^2)$ falls off like $1/Q^{2N-2M}$. Clearly the PTAs are a particular case of the PPAs, i.e. $\mathbb{T}_N^M(Q^2) = \mathbb{P}_{0,N}^M(Q^2)$ and coincide with what has been called the Hadronic Approximation to large- N_c QCD in the literature [13, 14].

Let us summarize the mathematical jargon. A Pade Type Approximant (PTA) is a rational function with all the poles chosen in advance precisely at the physical masses. A Pade Approximant (PA) is when all the poles are left free. The intermediate situation, with some poles fixed at the physical masses and some left free, corresponds to what is called a Partial Pade Approximant (PPA).

¹⁵For simplicity, we will assume that all the poles are simple.

3. Testing rational approximations

3.1 The model

Let us consider the two-point functions of vector and axial-vector currents in the chiral limit

$$\Pi_{\mu\nu}^{V,A}(q) = i \int d^4x e^{iqx} \langle J_\mu^{V,A}(x) J_\nu^{\dagger V,A}(0) \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi_{V,A}(q^2), \quad (3.1)$$

with $J_V^\mu(x) = \bar{d}(x)\gamma^\mu u(x)$ and $J_A^\mu(x) = \bar{d}(x)\gamma^\mu\gamma^5 u(x)$. As it is known, the difference $\Pi_V(q^2) - \Pi_A(q^2)$ satisfies an unsubtracted dispersion relation¹⁶

$$\Pi_{V-A}(q^2) = \int_0^\infty \frac{dt}{t - q^2 - i\epsilon} \frac{1}{\pi} \text{Im} \Pi_{V-A}(t). \quad (3.2)$$

Following refs. [31, 28], we define our model by giving the spectrum as

$$\begin{aligned} \frac{1}{\pi} \text{Im} \Pi_V(t) &= 2F_\rho^2 \delta(t - M_\rho^2) + 2 \sum_{n=0}^\infty F_V^2(n) \delta(t - M_V^2(n)), \\ \frac{1}{\pi} \text{Im} \Pi_A(t) &= 2F_0^2 \delta(t) + 2 \sum_{n=0}^\infty F_A^2(n) \delta(t - M_A^2(n)). \end{aligned} \quad (3.3)$$

Here F_ρ, M_ρ are the electromagnetic decay constant and mass of the ρ meson and $F_{V,A}(n)$ are the electromagnetic decay constants of the n -th resonance in the vector (resp. axial) channels, while $M_{V,A}(n)$ are the corresponding masses. F_0 is the pion decay constant in the chiral limit. The dependence on the resonance excitation number n is the following:

$$F_{V,A}^2(n) = F^2 = \text{constant}, \quad M_{V,A}^2(n) = m_{V,A}^2 + n \Lambda^2, \quad (3.4)$$

in accord with known properties of the large- N_c limit of QCD [5] as well as alleged properties of the associated Regge theory [36].

The combination

$$\Pi_{LR}(q^2) = \frac{1}{2}(\Pi_V(q^2) - \Pi_A(q^2)) \quad (3.5)$$

thus reads

$$\Pi_{LR}(q^2) = \frac{F_0^2}{q^2} + \frac{F_\rho^2}{-q^2 + M_\rho^2} + \sum_{n=0}^\infty \left\{ \frac{F^2}{-q^2 + M_V^2(n)} - \frac{F^2}{-q^2 + M_A^2(n)} \right\}. \quad (3.6)$$

This two-point function can be expressed in terms of the Digamma function $\psi(z) = \frac{d}{dz} \log \Gamma(z)$ as [28]

$$\Pi_{LR}(q^2) = \frac{F_0^2}{q^2} + \frac{F_\rho^2}{-q^2 + M_\rho^2} + \frac{F^2}{\Lambda^2} \left\{ \psi \left(\frac{-q^2 + m_A^2}{\Lambda^2} \right) - \psi \left(\frac{-q^2 + m_V^2}{\Lambda^2} \right) \right\}. \quad (3.7)$$

To resemble the case of QCD, we will demand that the usual parton-model logarithm is reproduced in both vector and axial-vector channels and that the difference (3.2) has an

¹⁶The upper cutoff which is needed to render the dispersive integrals mathematically well defined can be sent to infinity provided it respects chiral symmetry [17].

operator product expansion which starts at dimension six. A set of parameters satisfying these conditions is given by¹⁷

$$\begin{aligned}
 F_0 &= 85.8 \text{ MeV}, & F_\rho &= 133.884 \text{ MeV}, & F &= 143.758 \text{ MeV}, \\
 M_\rho &= 0.767 \text{ GeV}, & m_A &= 1.182 \text{ GeV}, & m_V &= 1.49371 \text{ GeV}, & \Lambda &= 1.2774 \text{ GeV},
 \end{aligned}
 \tag{3.8}$$

and is the one we will use in this section. This set of parameters has been chosen to resemble those of the real world, while keeping the model at a manageable level. For instance, the values of F_ρ and M_ρ in (3.8) are chosen so that the function Π_{LR} in (3.7) has vanishing $1/Q^2$ and $1/Q^4$ in the OPE at large $Q^2 > 0$, as in real QCD. In fact, the model admits the introduction of finite widths (which is a $1/N_c$ effect) in the manner described in ref. [31], after which the spectral function looks reasonably similar to the experimental spectral function. This comparison can be found in figure 5 of ref. [28]. But this model is also interesting for a very different reason. In ref. [28] several attempts were made at determining the coefficients of the OPE by using the methods which have become common practice in the literature. Among those we may list Finite Energy Sum Rules [32], with pinched weights [33], Laplace sum rules [34] and Minimal Hadronic Approximation [13, 14]. As it turned out, when these methods were tested on the model, none of them was able to produce very accurate results. We think that this makes the model very interesting (and challenging !) as a way to assess systematic errors [35].

Defining the expansion of the Green's function (3.2) in $Q^2 = -q^2$ around $Q^2 = \infty$ as

$$Q^2 \Pi_{LR}(-Q^2) \approx \sum_k C_{2k} Q^{2k}, \quad \text{with } k = -2, -3, \dots
 \tag{3.9}$$

one obtains that the coefficients accompanying inverse powers of momentum, akin to the Operator Product Expansion at large $Q^2 > 0$, are given by ($p = 1, 2, 3, \dots$ with $k = 1 - p$):

$$C_{2k} = -F_0^2 \delta_{p,1} + (-1)^{p+1} \left[F_\rho^2 M_\rho^{2p-2} - \frac{1}{p} F^2 \Lambda^{2p-2} \left\{ B_p \left(\frac{m_V^2}{\Lambda^2} \right) - B_p \left(\frac{m_A^2}{\Lambda^2} \right) \right\} \right],
 \tag{3.10}$$

where $B_p(x)$ are the Bernoulli polynomials [10]. As stated above, F_ρ and M_ρ are defined by the condition that the above expression (3.10) vanishes for $k = 0, -1$ (in agreement with (3.9) above), enforcing that $Q^2 \Pi_{LR}(-Q^2) \sim Q^{-4}$ at large momentum as in QCD. We emphasize that the above coefficients of the OPE in eq. (3.10) can not be calculated by a naive expansion at large Q^2 of the Green's function in eq. (3.6). In other words, physical masses and decay constants do not satisfy the naive Weinberg sum rules [17].

On the other hand, the expansion of the Green's function (3.2) in $Q^2 = -q^2$ around $Q^2 = 0$ can be written as

$$Q^2 \Pi_{LR}(-Q^2) \approx \sum_k C_{2k} Q^{2k}, \quad \text{with } k = 0, 1, 2, 3, \dots
 \tag{3.11}$$

¹⁷These numbers have been rounded off for the purpose of presentation. Some of the exercises which will follow require much more precision than the one shown here.

C_0	C_2	C_4	C_6	C_{-4}	C_{-6}	C_{-8}
-7.362	21.01	-43.92	81.81	-2.592	1.674	-0.577

Table 1: Values of the coefficients C_{2k} from the high- and low- Q^2 expansions of $Q^2 \Pi_{LR}(-Q^2)$ in eqs. (3.9), (3.11) in units of $10^{-3} GeV^{2-2k}$. Notice that $C_{-2} = 0$ and $C_0 = -F_0^2$ (the pion decay constant in the chiral limit), see text.

For these coefficients accompanying nonnegative powers of momentum, akin to the chiral expansion at small Q^2 , one has ($\ell = 1, 2, 3, \dots$):

$$C_0 = -F_0^2, \quad C_{2\ell} = (-1)^{\ell+1} \frac{F_\rho^2}{M_\rho^{2\ell}} - \frac{1}{(\ell-1)! \Lambda^{2\ell}} \left\{ \psi^{(\ell-1)} \left(\frac{m_V^2}{\Lambda^2} \right) - \psi^{(\ell-1)} \left(\frac{m_A^2}{\Lambda^2} \right) \right\}, \quad (3.12)$$

where $\psi^{(\ell-1)}(z) = d^{\ell-1}\psi(z)/dz^{\ell-1}$. In table 1 we collect the values for the first few of these coefficients C_{2k} for both expansions (3.9) and (3.11).

Following the definitions in section 2, let us start with the construction of the rational approximants to the function $Q^2 \Pi_{LR}(-Q^2)$ using as input the coefficients of its chiral expansion, eq. (3.12). Since our original function (3.7) falls off at large Q^2 as Q^{-4} , this is a constraint we will impose on all our rational approximants by selecting the appropriate difference in the order of the polynomials in the numerator and denominator.

3.2 Pade Approximants (PAs)

The simplest PA satisfying the right falloff at large momentum is $P_2^0(Q^2)$, so we will begin with this case. In order to simplify the results, and unless explicitly stated otherwise, we will assume that dimensionful quantities are expressed in units of GeV to the appropriate power. Fixing the three unknowns with the first three coefficients from the chiral expansion of (3.7) (i.e. $C_{0,2,4}$) one gets the following rational function

$$P_2^0(Q^2) = \frac{-r_R^2}{(Q^2 + z_R)(Q^2 + z_R^*)}, \quad r_R^2 = 3.379 \times 10^{-3}, \quad z_R = 0.6550 + i 0.1732. \quad (3.13)$$

We can hardly overemphasize the striking appearance of a pair of complex-conjugate poles on the Minkowski side of the complex Q^2 plane. Obviously, this means that these poles cannot be interpreted in any way as the meson states appearing in the physical spectrum (3.3), (3.6). In spite of this, if one expands (3.13) for large values of $Q^2 > 0$, one finds $C_{-4} = -r_R^2 = -3.379 \times 10^{-3}$ which is not such a bad approximation for this coefficient of the OPE, see table 1. Even better is the prediction of the fourth term in the chiral expansion, which is $C_6 = 79.58 \times 10^{-3}$.

This agreement is not a numerical coincidence and the approximation can be systematically improved if more terms of the chiral expansion are known. In order to exemplify this, we have amused ourselves by constructing the high-order PA $P_{52}^{50}(Q^2)$. This rational approximant correctly determines the values for $C_{-4,-6,-8}$ with (respectively) 52, 48 and 45 decimal figures. In the case of C_{103} , which is the first predictable term from the chiral expansion for this Pade, the accuracy reaches some staggering 192 decimal figures. This is all in agreement with Pommerenke's theorem [22].

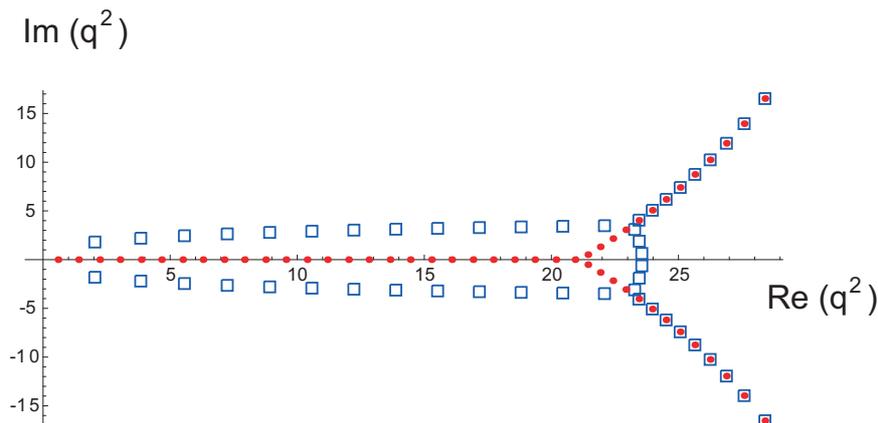


Figure 1: Location of the poles (dots) and zeros (squares) of the Pade Approximant $P_{52}^{50}(-q^2)$ in the complex q^2 plane. We recall that $Q^2 = -q^2$. Notice how zeros and poles approximately coincide in the region which is farthest away from the origin. When the order of the Pade is increased, the overall shape of the figure does not change but the two branches of complex poles move towards the right, i.e. away from the origin.

As it happens for the PA (3.13), also higher-order PAs may develop some artificial poles. In particular, figure 1 shows the location of the 52 poles of the PA $P_{52}^{50}(Q^2)$ in the complex q^2 plane. Of these, the first 25 are purely real and the rest are complex-conjugate pairs. A detailed numerical analysis reveals that the poles and residues reproduce very well the value of the meson masses and decay constants for the lowest part of the physical spectrum of the model given in (3.6), (3.8), but the agreement deteriorates very quickly as one gets farther away from the origin, eventually becoming the complex numbers seen in figure 1. It is by creating these analytic defects that rational functions can effectively mimic with a finite number of poles the infinite tower of poles present in the original function (3.7).

For instance the values of the first pole and residue in $P_{52}^{50}(Q^2)$ reproduce those of the ρ in (3.8) within 193 astonishing decimal places for both. However, in the case of the 25th pole, which is the last one still purely real, its location agrees with the physical mass only with 3 decimal figures. This is not to be considered as a success, however, because after the previous accuracy, this is quite a dramatic drop. In fact, the residue associated with this 25th pole comes out to be 29 times the true value. The lesson we would like to draw from this exercise should be clear: the determination of decay constants and masses extracted as the residues and poles of a PA deteriorate very quickly as one moves away from the origin. There is no reason why the last poles and residues in the PA are to be anywhere near their physical counterparts and their identification with the particle’s mass and decay constant should be considered unreliable. Clearly, this particularly affects low-order PAs.

A very good accuracy can also be obtained in the determination of global euclidean observables such as integrals of the Green’s function over the interval $0 \leq Q^2 < \infty$. Notice that the region where one approximates the true function is far away from the artificial

poles in the PA. For instance, one may consider the value for the integral

$$I_\pi = (-1) \int_0^\infty dQ^2 Q^2 \Pi_{LR}(Q^2) = 4.78719 \times 10^{-3}, \quad (3.14)$$

which, up to a constant, would yield the electromagnetic pion mass difference in the chiral limit [37] in the model (3.7). The PA $P_{52}^{50}(Q^2)$ reproduces the value for this integral with more than 42 decimal figures. This suggests that one may use the integral (3.14) as a further input to construct a PA.

For example if we fix the three unknowns in the PA $P_2^0(Q^2)$ by matching the first two terms from the chiral expansion but now we complete it with the pion mass difference (3.14) instead of a third term from the chiral expansion as we did in (3.13),¹⁸ the approximant results to be

$$\tilde{P}_2^0(Q^2) = \frac{-r_R^2}{(Q^2 + z_R)(Q^2 + z_R^*)}, \quad \text{with } r_R^2 = 2.898 \times 10^{-3}, \quad z_R = 0.5618 + i 0.2795. \quad (3.15)$$

This determines $C_{-4} = -2.898 \times 10^{-3}$ and $C_4 = -41.26 \times 10^{-3}$, which shows that using the pion mass difference is not a bad idea. Notice how the position of the artificial pole has changed with respect to (3.13).

Artificial poles and analytic defects are transient in nature, i.e. they appear and disappear from a point in the complex plane when the order of the Pade is changed. On the contrary, the typical sign that a pole in a Pade is associated with a truly physical pole is its stability under these changes in the order of the Pade. Of course, when the order in the Pade increases there have to be new poles by definition, and it is natural to expect that some of them will be defects. Pade Approximants place some effective poles and residues in the complex Q^2 plane in order to mimic the behavior of the true Green's function, but it can mimic the function only away from the poles, e.g. in the Euclidean region. Obviously, PAs cannot converge at the poles, in agreement with Pommerenke's theorem [22], since not even the true function is well defined there. The point is that what may look like a small correction in the Euclidean region may turn out to be a large number in the Minkowski region. To exemplify this in simple terms, let us consider a very small parameter ϵ and imagine that a given Pade $P(Q^2)$ produces the rational approximant to the true Green's function $G(Q^2)$ given by

$$G(Q^2) \approx P(Q^2) \equiv R(Q^2) + \frac{\epsilon}{Q^2 + M^2}, \quad (3.16)$$

where $R(Q^2)$ is the part of the Pade which is independent of ϵ . Although for $Q^2 > 0$ there is a sense in which the last term is a small correction precisely because of the smallness of ϵ , for $Q^2 < 0$ this is no longer true because of the pole at $Q^2 = -M^2$. This pole is in general a defect and may not represent any physical mass. In fact, associated with this pole, there is a very close-by zero of the Pade $P(Q^2)$ at $Q^2 = -M^2 - \epsilon R(-M^2)^{-1}$, as can be immediately checked in (3.16). This is another way of saying that a defect is characterized by having an

¹⁸We remark that this procedure, although reasonable from the phenomenological point of view, strictly speaking lies outside the standard mathematical theory of rational approximants [20, 21, 30].

abnormally small residue and is the origin of the pairs of zeros and poles in the y-shaped branches of figure 1. Therefore, not only are defects unavoidable but one could say they are even necessary for a Pade Approximant to approximate a meromorphic function with an infinite set of poles.

Similarly to masses, also decay constants may be unreliable. To see this, imagine now that our Pade is given by

$$P(Q^2) = \frac{F}{Q^2 + M^2} + \frac{\epsilon}{(Q^2 + M^2)(Q^2 + M^2 + \epsilon^2)}, \quad (3.17)$$

again for a very small ϵ . As before, the term proportional to ϵ may be considered a small correction for $Q^2 > 0$. However, at the pole $Q^2 = -M^2$ the decay constant becomes $F + \epsilon^{-1}$ which, for ϵ small, may represent a huge correction. When the poles are preassigned at the physical masses, like in the case of PTAs, it is the value of the residues that compensates for the fact that the rational approximant lacks the infinite tower of resonances. As we saw before, the residues of the poles in the Pade which lie farthest away from the origin are the ones which get the largest distortion relative to their physical counterparts.

In real life, the number of available terms from the chiral expansion for the construction of a PA is very limited. Since the masses and decay constants of the first few vector and axial-vector resonances are known, one may envisage the construction of a rational approximant having some of its poles at the prescribed values given by the known masses of these resonances. If all the poles in the approximant are prescribed this way (as in the MHA), we have a PTA. On the contrary, when some of the poles are prescribed but some are also left free, then we have a PPA (see the previous section).

3.3 Pade Type Approximants (PTAs)

Assuming that the first masses are known, let us proceed to constructing the PTAs (2.6). The lowest such PTA is $\mathbb{T}_2^0(Q^2)$, which contains two poles at the physical masses of the ρ and the first A in the tower. Fixing the residue through the chiral expansion to be $C_0 = -F_0^2$, one obtains

$$\mathbb{T}_2^0(Q^2) = \frac{-F_0^2 M_\rho^2 M_A^2}{(Q^2 + M_\rho^2)(Q^2 + M_A^2)}. \quad (3.18)$$

Even though it has the same number of inputs (C_0 and the two masses), this rational approximant does not do such a good job as the PAs (3.13) or (3.15). For instance, C_{-4} is 2.3 times larger than the true value in table 1. As we have already stated, one way to intuitively understand this result is the following. The OPE is an expansion at $Q^2 = \infty$ and therefore knows about the whole spectrum because no resonance is heavy enough with respect to Q^2 to become negligible in the expansion, i.e., the infinite tower of resonances does not decouple in the OPE. Chopping an infinite set of poles down to a finite set may be a good approximation, but only at the expense of some changes. These changes amount to the appearance of poles and residues in the PA which the original function does not have. This is how the PA (3.13) manages to approximate the true function (3.7). However, by construction, the PTA (3.18) does not allow the presence of any artificial pole

because, unlike in a PA, all its poles are fixed at the physical values. Consequently, it only has its residues as a means to compensate for the infinite tower of poles present in the true function and, hence, does a poorer job than the PA (3.13), particularly in determining large- Q^2 observables like C_{-4} . Indeed, the role played by the residues in the approximation can be appreciated by comparing the true values of the decay constants to those extracted from (3.18). Although the one of the ρ is within 30% of the true value, that of the A is off by 100%.

A different matter is the prediction of low-energy observables such as, e.g., the chiral coefficients. In this case heavy resonances make a small contribution and this means that the infinite tower of resonances does decouple.¹⁹ Truncating the infinite tower down to a finite set of poles is not such a severe simplification in this case, which helps understand why a PTA may do a good job predicting unknown chiral coefficients. Indeed, (3.18) reproduces the value of C_2 within an accuracy of 15%, growing to 22% in the case of C_4 . A global observable like I_π averages the low and the high Q^2 behaviors and ends up differing from the true value (3.14) by 35%. This gives some confidence that observables which are integrals over Euclidean momentum may be reasonably estimated with MHA as, e.g., in the B_K calculation of ref. [38].

Improving on the PTA (3.18) by adding in the first resonance mass from the vector tower produces the following approximant

$$\mathbb{T}_3^1(Q^2) = \frac{a + b Q^2}{(Q^2 + M_\rho^2)(Q^2 + M_A^2)(Q^2 + M_V^2)}, \text{ with } \begin{cases} a = -13.5 \times 10^{-3}, \\ b = +1.33 \times 10^{-4} \end{cases}, \quad (3.19)$$

where the values of the chiral coefficients C_0 and C_2 have been used to determine the parameters a and b . The prediction for C_4 is much better now (only 2% off), in agreement with our previous comments. The prediction for C_{-4} is still very bad, becoming now 19 times smaller than the exact value. Nevertheless, it eventually gets much better if PTAs of very high order are constructed. For instance, we have found $C_{-4} = -2.58 \times 10^{-3}$ for the approximant \mathbb{T}_9^7 with 9 poles. Similarly, we have also checked that the prediction of the chiral coefficients and the integral (3.14) improve with higher-order PTAs.

However, another matter is the prediction of the residues. For instance, the prediction for the decay constant of the state with mass M_V in (3.19) is smaller than the exact value in the model (3.8) by a factor of 2. In general, we have seen that the residues of the poles always deteriorate very quickly so that the residue corresponding to the pole which is at the greatest distance from the origin is nowhere near the exact value. We again explicitly checked this up to the approximant \mathbb{T}_9^7 , in which case the decay constant for this pole is almost 5 times smaller than the exact value. The conclusion, therefore, is that PTAs are able to approximate the exact function only at the expense of changing the residues of the poles from their physical values. Identifying residues with physical decay constants may be completely wrong in a PTA for the poles which are farthest away from the origin.

¹⁹This is because the residues F^2 in the Green's function (3.7) stay constant as the masses grow. This behavior does not hold in the case of the scalar and pseudoscalar two-point functions [10].

3.4 Partial Padé Approximants (PPAs)

As an intermediate approach between PAs and PTAs, there are the PPAs (2.3) where some poles are fixed at their physical values while some others are left free. The simplest of such rational approximants is $\mathbb{P}_{1,1}^0(Q^2)$ (see the previous section for notation). Fixing its 3 unknowns with M_ρ^2, C_0 and C_2 , one obtains

$$\mathbb{P}_{1,1}^0(Q^2) = \frac{-r_R^2}{(Q^2 + M_\rho^2)(Q^2 + z_R)}, \quad \text{with } r_R^2 = 3.75 \times 10^{-3}, \quad z_R = 0.8665. \quad (3.20)$$

As can be seen, the mass (squared) of the first A resonance is predicted to be at z_R which is sensibly smaller than the true value in (3.8).²⁰ The rational function (3.20) predicts $C_{-4} = -r_R^2 = -3.75 \times 10^{-3}$ which is a better determination than that of the PTA (3.18) with the same number of inputs, and $C_4 = -45.52 \times 10^{-3}$ which is not bad either. Concerning the pion mass difference, one gets $I_\pi = 5.22 \times 10^{-3}$. However, as compared to the PAs (3.13) or (3.15), the PPA (3.20) does not represent a clear improvement.

In order to improve on accuracy of the PPA, one may try to use the mass and decay constant of the first resonance, M_ρ and F_ρ , in addition to the pion mass difference and the chiral coefficients C_0, C_2 and build the $\mathbb{P}_{2,1}^1(Q^2)$, which can be written as:

$$\mathbb{P}_{2,1}^1(Q^2) = \frac{F_\rho^2 M_\rho^2}{Q^2 + M_\rho^2} + \frac{a - F_\rho^2 M_\rho^2 Q^2}{(Q^2 + z_c)(Q^2 + z_c^*)}, \quad \begin{cases} a = 17.43 \times 10^{-3}, \\ z_c = 1.24 + i 0.34 \end{cases}. \quad (3.21)$$

This PPA, upon expansion at large and small Q^2 , determines $C_{-4} = -2.47 \times 10^{-3}$ and $C_4 = -44.0 \times 10^{-3}$ to be compared with the corresponding coefficient in table 1. The accuracy obtained is better than that of (3.15), but this is probably to be expected since (3.21) has more inputs.

Based on the previous numerical experiments done on the model in eq. (3.7), (3.8) (and many others), we now summarize the following conclusions. Although, in principle, the PAs have the advantage of reaching the best precision by carefully adjusting the polynomial in the denominator to have some effective poles which simulate the infinite tower present in (3.7), they have the disadvantage that some of the terms in the low- Q^2 expansion are required precisely to construct this denominator. This hampers the construction of high-order PAs and consequently limits the possible accuracy.

When the locations of the first poles in the true function are known, there is the possibility to construct PTAs (with all the poles fixed at the true values) and PPA (with some of the poles fixed and some left free). As we have seen, although the PTA may approximate low- Q^2 properties of the true function reasonably well, the large- Q^2 properties tend to be much worse, at least as long as they are not of unrealistically high order. The PPAs, on the other hand, interpolate smoothly between the PAs (only free poles) and the PTAs (no free pole). Depending on the case, one may choose one or several of these rational approximants. However, common to all the rational approximants constructed is the fact that the residues and/or poles which are farthest away from the origin are in general unrelated to their physical counterparts.

²⁰Intriguingly enough, this is also what happens in the real case of QCD [9, 19].

4. The QCD case

Let us now discuss the real case of large- N_c QCD in the chiral limit. In contrast to the case of the previous model, any analysis in this case is limited by two obvious facts. First, any input value will have an error (from experiment and because of the chiral and large- N_c limits), and this error will propagate through the rational approximant. And second, it is not possible to go to high orders in the construction of rational approximants due to the rather sparse set of input data. In spite of these difficulties one may feel encouraged by the phenomenological fact that resonance saturation approximates meson physics rather well.

The simplest PA to the function $Q^2\Pi_{LR}(-Q^2)$ with the right fall-off as Q^{-4} at large Q^2 is $P_2^0(Q^2)$:

$$P_2^0(Q^2) = \frac{a}{1 + A Q^2 + B Q^4} . \quad (4.1)$$

The values of the three unknowns a, A and B may be fixed by requiring that this PA reproduces the correct values for F_0, L_{10} and I_π ²¹ given by

$$\begin{aligned} F_0 &= 0.086 \pm 0.001 \text{ GeV} , \\ \delta m_\pi &= 4.5936 \pm 0.0005 \text{ MeV} \implies I_\pi = (5.480 \pm 0.006) \times 10^{-3} \text{ GeV}^4 , \\ L_{10}(0.5 \text{ GeV}) \leq L_{10} \leq L_{10}(1.1 \text{ GeV}) &\implies L_{10} = (-5.13 \pm 0.6) \times 10^{-3} . \end{aligned} \quad (4.2)$$

The low-energy constant L_{10} is related to the chiral coefficient C_2 , in the notation of eq. (3.11), by $C_2 = -4L_{10}$. Since L_{10} does not run in the large- N_c limit, it is not clear at what scale to evaluate $L_{10}(\mu)$ [39]. In eq. (4.2) we have varied μ in the range $0.5 \text{ GeV} \leq \mu \leq 1.1 \text{ GeV}$ as a way to estimate $1/N_c$ systematic effects. The central value corresponds to the result for $L_{10}(M_\rho)$ found in ref. [41]. The other results in (4.2) are extracted from refs. [1, 40].

Obviously, the PA (4.1) can also be rewritten as

$$P_2^0(Q^2) = \frac{-r^2}{(Q^2 + z_V)(Q^2 + z_A)} , \quad (4.3)$$

in terms of two poles $z_{V,A}$. In order to discuss the nature of these poles, we will define the dimensionless parameter ζ by the combination

$$\zeta \equiv -4L_{10} \frac{I_\pi}{F_0^4} = 2.06 \pm 0.25 , \quad (4.4)$$

where the values in (4.2) above have been used in the last step. Imposing the constraints (4.2) on the PA (4.3) one finds two types of solutions depending on the value of ζ : for $\zeta > 2$ the two poles $z_{V,A}$ are real, whereas for $\zeta < 2$ the two poles are complex. At $\zeta = 2$, the two solutions coincide. To see this, let us write the set of equations satisfied

²¹Recall that I_π is, up to a constant, the electromagnetic pion mass difference δm_π [19] and is defined in terms of Π_{LR} as in eq. (3.14).

C_0	C_2	C_4	C_6	C_8	C_{-4}	C_{-6}	C_{-8}
$-F_0^2$	$-4L_{10}$	-43 ± 13	81 ± 53	-145 ± 120	-4.1 ± 0.5	6 ± 2	-7 ± 6

Table 2: Values of the coefficients C_{2k} in the high- and low- Q^2 expansions of $Q^2 \Pi_{LR}(-Q^2)$ in eq. (3.9), (3.11) in units of $10^{-3} GeV^{2-2k}$. Recall that $C_{-2} = 0$.

by the PA (4.3) as:

$$\begin{aligned}
 F_0^2 &= \frac{r^2}{z_V z_A} \\
 -4L_{10} &= F_0^2 \left(\frac{1}{z_V} + \frac{1}{z_A} \right) \\
 I_\pi &= F_0^2 \frac{z_V z_A}{z_A - z_V} \log \frac{z_A}{z_V}.
 \end{aligned}
 \tag{4.5}$$

The first of these equations can be used to determine the value of the residue r^2 in terms of $z_V z_A$. In order to analyze the other two, let us first assume that both poles $z_{V,A}$ are real. In this case, they also have to be positive or else the integral I_π will not exist because it runs over all positive values of Q^2 . Let us now make the change of variables

$$z_V = R(1-x), \quad z_A = R(1+x) \quad . \tag{4.6}$$

The condition $z_{V,A} > 0$ translates into $R > 0, |x| < 1$. In terms of these new variables, the second and third equations in (4.5) can be combined into

$$\zeta = \frac{1}{x} \log \frac{1+x}{1-x}, \tag{4.7}$$

where the definition (4.4) for ζ has been used. With the help of the identity $\log(1+x/1-x) = 2 \operatorname{th}^{-1}x$ (for $|x| < 1$), one can finally rewrite this expression as

$$\zeta = \frac{2}{x} \operatorname{th}^{-1}x, \quad (x \text{ real}) \tag{4.8}$$

which is an equation with a solution for x only if $\zeta \geq 2$. Once this value of x is found, the value of R can always be obtained from one of the last two equations (4.5) and this determines the two real poles $z_{V,A}$ from (4.6).

On the other hand, when $\zeta < 2$, eq. (4.8) does not have a solution. However, according to (4.4), ζ can *also* be smaller than 2. In order to study this case, we may use the identity $\operatorname{th}^{-1}(iy) = i \tan^{-1}(y)$ to rewrite the above equation (4.8) in terms of the variable $x = iy$ (y real) as

$$\zeta = \frac{2}{y} \tan^{-1}y, \quad (y \text{ real}). \tag{4.9}$$

One now finds that this equation has a solution for y when $\zeta \leq 2$. In this case the poles of the PA (4.1) are complex-conjugate to each other and can be obtained as $z_{V,A} = R(1 \pm iy)$. These poles, obviously, cannot be associated with any resonance mass and this is why this solution has been discarded in all resonance saturation schemes up to now. However, from the point of view of the rational approximant (4.1) there is nothing wrong with this

complex solution, as the approximant is real and well behaved. From the lessons learned in the previous section with the model, there is no reason to discard this solution since, as we saw, rational approximants may use complex poles to produce accurate approximations. Therefore, we propose to use *both* the complex as well as the real solution for the poles $z_{V,A}$, at least insofar as the value for $\zeta \gtrsim 2$. In this case we obtain, using the values given in eqs. (4.2),

$$(\zeta \geq 2), \quad r^2 = -(4.1 \pm 0.5) \times 10^{-3}, \quad z_V = (0.77)^2 \pm 0.15, \quad z_A = (0.96)^2 \pm 0.41 \quad (4.10)$$

$$(\zeta \leq 2), \quad r^2 = -(3.9 \pm 0.1) \times 10^{-3}, \quad z_V = z_A^* = (0.66 \pm 0.06) + i(0.25 \pm 0.25), \quad (4.11)$$

in units of GeV^6 for r^2 , and GeV^2 for $z_{V,A}$. The two solutions in eqs. (4.10), (4.11) have been separated for illustrative purposes only. It is clear that they are continuously connected through the boundary at $\zeta = 2$, at which value the two poles coincide and $z_V = z_A \simeq 0.72$. The errors quoted are the result of scanning the spread of values in (4.2) through the equations (4.5).

With both set of values in (4.10), (4.11), one can get to a prediction for the chiral and OPE coefficients by expansion in Q^2 and $1/Q^2$, respectively. These expansions of the PA can be done entirely in the Euclidean region $Q^2 > 0$, away from the position of the poles $z_{V,A}$, whether real or complex. Recalling the notation in eq. (3.9), (3.11), the above $P_2^0(Q^2)$ produces the coefficients for these expansions collected in table 2. The values for the OPE coefficients $C_{-4,-6,-8}$ in this table are compatible with those of ref. [19], after multiplying by a factor of two in order to agree with the normalization used by these authors. However, the spectrum in our case is different because of the complex solution in (4.11). As we saw in the previous section with a model, this again shows that Euclidean properties of a given Green's function, such as the OPE and chiral expansions, or integrals over $Q^2 > 0$ are safer to approximate with a rational approximant than Minkowskian quantities, such as resonance masses and decay constants.

5. Conclusions

In this article we pointed out that approximating large- N_c QCD with a finite number of resonances may be reinterpreted within the mathematical Theory of Pade Approximants to meromorphic functions [20, 21].

The main results of this theory may be summarized as follows. One may expect convergence of a sequence of Pade Approximants to a QCD Green's function in the large- N_c limit in any compact region of the complex Q^2 plane except at most in a zero-area set [22]. This set without convergence comprises the poles of the original Green's function together with some other artificial poles generated by the approximant which the original function does not have. As the order of the PA grows, the previous convergence property implies that any given artificial pole either goes to infinity, away from the relevant region, or is almost compensated by a nearby zero. This symbiosis between a pole and a zero is called a defect. Although close to a pole the rational approximation breaks down, in a region which is far away from it the approximation should work well.

We have reviewed the main results of this theory with the help of a model for the two-point Green's function $\langle VV - AA \rangle$. The simpler case of a Green's function of the Stieltjes type, such as the two-point correlator $\langle VV \rangle$, was previously considered in ref. [23]. We have seen in the case of this particular model how rational approximants create the expected artificial poles (and the corresponding residues) in the Minkowski region $\text{Re}(q^2) > 0$ while, at the same time, yielding an accurate description of the Green's function in the Euclidean region $\text{Re}(q^2) < 0$. This happens in a hierarchical way: although the first poles/residues in a PA may be used to describe the physical masses/decay constants reasonably well, the last ones give only a very poor description. Therefore, it is in general unreliable to extract properties of individual mesons, such as masses and decay constants, from an approximation to large- N_c QCD with only a finite number of states. Since a form factor, like a decay constant, is obtained as the residue of a Green's function at the corresponding pole(s), this also means that one may not extract a meson form factor from a rational approximant to a 3-point Green's function, in agreement with [18]. This observation may explain why the analysis of ref. [42], which is based on an extraction of matrix elements such as $\langle \pi|S|P \rangle$ and $\langle \pi|P|S \rangle$ from the 3-point function $\langle SPP \rangle$, finds values for the $K_{\ell 3}$ form factor which are different from those obtained in other analyses [43].

In spite of all the above problems related to the Minkowski region, our model shows how Pade Approximants may nevertheless be a useful tool in other regions of momentum space. We think that this is also true in the real case of QCD in the large- N_c limit. In this case one may use the first few terms of the chiral and operator product expansions of a given Green's function to construct a Pade Approximant which should yield a reasonable description of this function in those regions of momentum space which are free of poles. In this construction, Pade Approximants containing complex poles, if they appear, should not be dismissed.

We have also reanalyzed the simplest approximation to the $\langle VV - AA \rangle$ Green's function in real QCD which consists of keeping only two poles, and we have found that, depending on the value of the combination ζ in eq. (4.4), these two poles may actually be complex.

However, if not all the residues and masses in a rational approximant are physical, this poses a challenge to any attempt to use a Lagrangian with a finite number of resonances such as, for example, the ones in ref. [9, 11], for describing Green's functions in the large- N_c limit of QCD. Even if these Lagrangians are interpreted in terms of PTAs, with the poles fixed at the physical value of the meson masses, we have seen how the residues then get very large corrections with respect to their physical counterparts. Can these residues be efficiently incorporated in a Lagrangian framework? We hope to be able to devote some work to answering this and related questions in the future.

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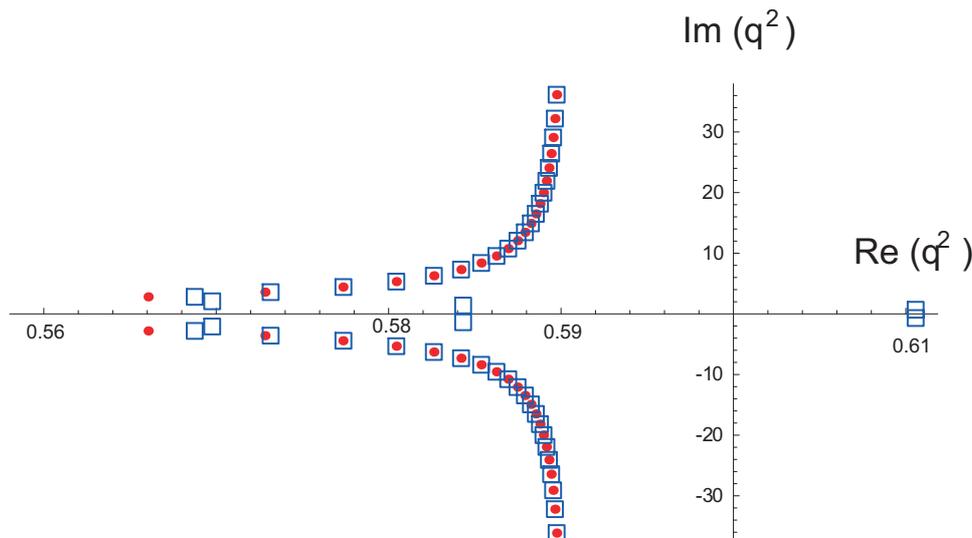


Figure 2: Location of the poles (dots) and zeros (squares) of the Pade Approximant $P_{50}^{50}(-1/q^2)$, constructed from the OPE of $Q^2\Pi_{LR}$ in eqs. (3.9), (3.10), in the complex q^2 plane. We recall that $Q^2 = -q^2$. The poles are all complex-conjugate pairs.

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A. OPE-based Pade Approximant

Here, unlike in the rest of the paper, we will deal with the construction of PAs from the $1/Q^2$ expansion at infinity. We will show how the PAs constructed from this expansion (akin to the OPE) do not in general reproduce even the first resonances in the spectrum, unlike those constructed from the chiral expansion. Again, we will use the model of section 3 as an example. Recalling the definition of the OPE given in eq. (3.9), with the corresponding coefficients (3.10), it is straightforward to construct a PA in $1/Q^2$ around infinity, i.e. by matching powers of the OPE in $1/Q^2$. The construction parallels that in eq. (2.2) but with the replacement $z = 1/Q^2$. Since the function $Q^2\Pi_{LR}(-Q^2)$ behaves like a constant for $Q^2 \rightarrow 0$, we will consider diagonal Pade Approximants, i.e. of the form $P_n^n(1/Q^2)$, in order to reproduce this behavior. Figure 2 shows the position of the poles and zeros of the PA $P_{50}^{50}(-1/q^2)$ in the complex q^2 plane. As it is clear from this plot, the positions of the poles have nothing to do with the physical masses in the model, given by eqs. (3.4), (3.8), even for the lightest states. This is to be contrasted with what happens with the PA constructed from the chiral expansion around Q^2 , which is shown in figure 1. The difference between the two behaviors is due to the fact that, while the chiral expansion has a finite radius of convergence, the radius of convergence of the OPE vanishes because this expansion is asymptotic. In spite of this, one can see that the PA $P_{50}^{50}(1/Q^2)$ is an excellent approximation to the function $Q^2\Pi_{LR}(-Q^2)$ in the euclidean region.

References

- [1] J. Gasser and H. Leutwyler, *Chiral perturbation theory: expansions in the mass of the strange quark*, *Nucl. Phys. B* **250** (1985) 465.
- [2] H.W. Fearing and S. Scherer, *Extension of the chiral perturbation theory meson lagrangian to order $p(6)$* , *Phys. Rev. D* **53** (1996) 315 [[hep-ph/9408346](#)];
J. Bijnens, G. Colangelo and G. Ecker, *The mesonic chiral lagrangian of order p^6* , *JHEP* **02** (1999) 020 [[hep-ph/9902437](#)];
J. Bijnens, L. Girlanda and P. Talavera, *The anomalous chiral lagrangian of order p^6* , *Eur. Phys. J. C* **23** (2002) 539 [[hep-ph/0110400](#)].
- [3] J. Kambor, J. Missimer and D. Wyler, *The chiral loop expansion of the nonleptonic weak interactions of mesons*, *Nucl. Phys. B* **346** (1990) 17;
G. Esposito-Farese, *Right invariant metrics on SU(3) and one loop divergences in chiral perturbation theory*, *Z. Physik C* **50** (1991) 255.
- [4] C. Bernard et al., *Low energy constants from the MILC collaboration*, [hep-lat/0611024](#) and references therein.
- [5] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys. B* **72** (1974) 461;
E. Witten, *Baryons in the $1/N$ expansion*, *Nucl. Phys. B* **160** (1979) 57.
- [6] S.R. Coleman and E. Witten, *Chiral symmetry breakdown in large- N chromodynamics*, *Phys. Rev. Lett.* **45** (1980) 100.
- [7] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)].
- [8] J.J. Sakurai, *Currents and mesons*, Chicago Lectures in Physics, The Univ. of Chicago Press (1969);
see also A. Bramon, E. Etim and M. Greco, *A vector meson dominance approach to scale invariance*, *Phys. Lett. B* **41** (1972) 609.
- [9] J.F. Donoghue, C. Ramirez and G. Valencia, *The spectrum of QCD and chiral lagrangians of the strong and weak interactions*, *Phys. Rev. D* **39** (1989) 1947;
G. Ecker, J. Gasser, A. Pich and E. de Rafael, *The role of resonances in chiral perturbation theory*, *Nucl. Phys. B* **321** (1989) 311.
- [10] M. Golterman and S. Peris, *On the relation between low-energy constants and resonance saturation*, *Phys. Rev. D* **74** (2006) 096002 [[hep-ph/0607152](#)].
- [11] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, *Chiral lagrangians for massive spin 1 fields*, *Phys. Lett. B* **223** (1989) 425;
for an update, see ref. [14]. See also J. Bijnens and E. Pallante, *On the tensor formulation of effective vector lagrangians and duality transformations*, *Mod. Phys. Lett. A* **11** (1996) 1069 [[hep-ph/9510338](#)].
- [12] B. Moussallam, *A sum rule approach to the violation of Dashen's theorem*, *Nucl. Phys. B* **504** (1997) 381 [[hep-ph/9701400](#)].
- [13] M. Knecht and E. de Rafael, *Patterns of spontaneous chiral symmetry breaking in the large- N_c limit of QCD-like theories*, *Phys. Lett. B* **424** (1998) 335 [[hep-ph/9712457](#)];
S. Peris, M. Perrottet and E. de Rafael, *Matching long and short distances in large- N_c QCD*, *JHEP* **05** (1998) 011 [[hep-ph/9805442](#)].

- [14] A. Pich, *Present status of chiral perturbation theory*, *Int. J. Mod. Phys. A* **20** (2005) 1613 [hep-ph/0410322].
- [15] S. Peris, *Electroweak matrix elements at large- N_c : matching quarks to mesons*, hep-ph/0204181;
E. de Rafael, *Analytic approaches to kaon physics*, *Nucl. Phys.* **119** (Proc. Suppl.) (2003) 71 [hep-ph/0210317].
- [16] S. Peris, M. Perrottet and E. de Rafael, *Two loop electroweak corrections to the muon $G-2$: a new class of hadronic contributions*, *Phys. Lett.* **B 355** (1995) 523 [hep-ph/9505405];
M. Knecht, S. Peris, M. Perrottet and E. de Rafael, *Decay of pseudoscalars into lepton pairs and large- N_c QCD*, *Phys. Rev. Lett.* **83** (1999) 5230 [hep-ph/9908283];
M.F.L. Golterman and S. Peris, *The 7/11 rule: an estimate of m_ρ/f_π* , *Phys. Rev.* **D 61** (2000) 034018 [hep-ph/9908252];
M. Knecht, S. Peris and E. de Rafael, *A critical reassessment of $Q(7)$ and $Q(8)$ matrix elements*, *Phys. Lett.* **B 508** (2001) 117 [hep-ph/0102017];
M. Knecht and A. Nyffeler, *Resonance estimates of $O(p^6)$ low-energy constants and QCD short-distance constraints*, *Eur. Phys. J.* **C 21** (2001) 659 [hep-ph/0106034];
M. Knecht, S. Peris, M. Perrottet and E. De Rafael, *Electroweak hadronic contributions to $G_\mu-2$* , *JHEP* **11** (2002) 003 [hep-ph/0205102];
M. Golterman and S. Peris, *Analytic estimates for penguin operators in quenched QCD*, *Phys. Rev.* **D 68** (2003) 094506 [hep-lat/0306028];
T. Hambye, S. Peris and E. de Rafael, $\Delta_i = 1/2$ and e'/ϵ in large- N_c QCD, *JHEP* **05** (2003) 027 [hep-ph/0305104];
V. Cirigliano, G. Ecker, M. Eidemuller, A. Pich and J. Portoles, *The $\langle VAP \rangle$ Green function in the resonance region*, *Phys. Lett.* **B 596** (2004) 96 [hep-ph/0404004];
J. Bijnens, E. Gamiz and J. Prades, *Hadronic matrix elements for kaons*, *Nucl. Phys.* **133** (Proc. Suppl.) (2004) 245 [hep-ph/0309216];
V. Cirigliano et al., *The $\langle SPP \rangle$ Green function and SU(3) breaking in K_{l3} decays*, *JHEP* **04** (2005) 006 [hep-ph/0503108]; *Towards a consistent estimate of the chiral low-energy constants*, *Nucl. Phys.* **B 753** (2006) 139 [hep-ph/0603205];
J. Bijnens, E. Gamiz and J. Prades, *The B_K kaon parameter in the chiral limit*, *JHEP* **03** (2006) 048 [hep-ph/0601197].
- [17] M. Golterman and S. Peris, *On the use of the operator product expansion to constrain the hadron spectrum*, *Phys. Rev.* **D 67** (2003) 096001 [hep-ph/0207060].
- [18] J. Bijnens, E. Gamiz, E. Lipartia and J. Prades, *QCD short-distance constraints and hadronic approximations*, *JHEP* **04** (2003) 055 [hep-ph/0304222].
- [19] S. Friot, D. Greynat and E. de Rafael, *Chiral condensates, $Q(7)$ and $Q(8)$ matrix elements and large- N_c QCD*, *JHEP* **10** (2004) 043 [hep-ph/0408281].
- [20] G.A. Baker and P. Graves-Morris, *Pade approximants*, Encyclopedia of Mathematics and its Applications, Cambridge Univ. Press (1996).
- [21] G.A. Baker, *Essentials of pade approximants*, Academic Press (1975).
- [22] C. Pommerenke, *Pade approximants and convergence in capacity*, *J. Math. Anal. Appl.* **41** (1973) 775;
Reviewed in ref. [20], Section 6.5, Theorem 6.5.4, Collorary 1.
- [23] S. Peris, *Large- N_c QCD and pade approximant theory*, *Phys. Rev.* **D 74** (2006) 054013 [hep-ph/0603190].

- [24] A.A. Migdal, *Multicolor QCD as dual resonance theory*, *Ann. Phys. (NY)* **109** (1977) 365.
- [25] O. Cata, *Migdal's model and holography*, [hep-ph/0701196](#); *Migdal's model and holography*, [hep-ph/0701196](#); *Towards understanding Regge trajectories in holographic QCD*, [hep-ph/0605251](#).
- [26] See, e.g., M. Golterman and S. Peris, *Large- N_c QCD meets Regge theory: the example of spin-one two-point functions*, *JHEP* **01** (2001) 028 [[hep-ph/0101098](#)]; A.A. Andrianov, S.S. Afonin, D. Espriu and V.A. Andrianov, *Matching meson resonances to ope in QCD*, *Int. J. Mod. Phys. A* **21** (2006) 885 [[hep-ph/0509144](#)].
- [27] J. Erlich, G.D. Kribs and I. Low, *Emerging holography*, *Phys. Rev. D* **73** (2006) 096001 [[hep-th/0602110](#)] and references therein; A. Falkowski and M. Perez-Victoria, *Holography, pade approximants and deconstruction*, *JHEP* **02** (2007) 086 [[hep-ph/0610326](#)].
- [28] O. Cata, M. Golterman and S. Peris, *Duality violations and spectral sum rules*, *JHEP* **08** (2005) 076 [[hep-ph/0506004](#)].
- [29] Discussion session led by J. Donoghue at the workshop *Matching light quarks to hadrons*, Benasque Center for Science, Benasque, Spain, July-August 2004, <http://benasque.ecm.ub.es/2004quarks/2004quarks.htm>.
- [30] C. Diaz-Mendoza, P. Gonzalez-Vera and R. Orive, *On the convergence of two-point partial Padé approximants for meromorphic functions of Stieltjes type*, *Appl. Num. Math.* **53** (2005) 39 and references therein.
- [31] M.A. Shifman, *Quark-hadron duality*, [hep-ph/0009131](#).
- [32] J. Rojo and J.I. Latorre, *Neural network parametrization of spectral functions from hadronic tau decays and determination of QCD vacuum condensates*, *JHEP* **01** (2004) 055 [[hep-ph/0401047](#)]; J. Bijnens, E. Gamiz and J. Prades, *Matching the electroweak penguins $Q(7)$, $Q(8)$ and spectral correlators*, *JHEP* **10** (2001) 009 [[hep-ph/0108240](#)]; S. Peris, B. Phily and E. de Rafael, *Tests of large- N_c QCD from hadronic tau decay*, *Phys. Rev. Lett.* **86** (2001) 14 [[hep-ph/0007338](#)].
- [33] V. Cirigliano, E. Golowich and K. Maltman, *QCD condensates for the light quark V-A correlator*, *Phys. Rev. D* **68** (2003) 054013 [[hep-ph/0305118](#)]; C.A. Dominguez and K. Schilcher, *Finite energy chiral sum rules in QCD*, *Phys. Lett. B* **581** (2004) 193 [[hep-ph/0309285](#)].
- [34] S. Narison, *V-a hadronic tau decays: a laboratory for the QCD vacuum*, *Phys. Lett. B* **624** (2005) 223 [[hep-ph/0412152](#)].
- [35] A.A. Almasy, K. Schilcher and H. Spiesberger, *QCD condensates of dimension $D = 6$ and $D = 8$ from hadronic tau decays*, [hep-ph/0612304](#).
- [36] G. 't Hooft, *A two-dimensional model for mesons*, *Nucl. Phys. B* **75** (1974) 461; C.G. Callan Jr., N. Coote and D.J. Gross, *Two-dimensional Yang-Mills theory: a model of quark confinement*, *Phys. Rev. D* **13** (1976) 1649; M.B. Einhorn, *Form-factors and deep inelastic scattering in two-dimensional quantum chromodynamics*, *Phys. Rev. D* **14** (1976) 3451.
- [37] M. Knecht, S. Peris and E. de Rafael, *The electroweak $\pi^+\pi^0$ mass difference and weak matrix elements in the $1/N_c$ expansion*, *Phys. Lett. B* **443** (1998) 255 [[hep-ph/9809594](#)].

- [38] S. Peris and E. de Rafael, $K^0\bar{K}^0$ mixing in the $1/N_c$ expansion, *Phys. Lett.* **B 490** (2000) 213 [[hep-ph/0006146](#)];
O. Cata and S. Peris, *Kaon mixing and the charm mass*, *JHEP* **07** (2004) 079 [[hep-ph/0406094](#)]; *Long-distance dimension-eight operators in B_K* , *JHEP* **03** (2003) 060 [[hep-ph/0303162](#)].
- [39] O. Cata and S. Peris, *An example of resonance saturation at one loop*, *Phys. Rev.* **D 65** (2002) 056014 [[hep-ph/0107062](#)];
I. Rosell, P. Ruiz-Femenia and J. Portoles, *One-loop renormalization of resonance chiral theory: scalar and pseudoscalar resonances*, *JHEP* **12** (2005) 020 [[hep-ph/0510041](#)];
J. Portoles, I. Rosell and P. Ruiz-Femenia, *Vanishing chiral couplings in the large- N_c resonance theory*, [hep-ph/0611375](#); *Vanishing chiral couplings in the large- N_c resonance theory*, [hep-ph/0611375](#).
- [40] PARTICLE DATA GROUP, W.M. Yao et al., *Review of particle physics*, *J. Phys.* **G 33** (2006) 1.
- [41] M. Davier, L. Girlanda, A. Hocker and J. Stern, *Finite energy chiral sum rules and tau spectral functions*, *Phys. Rev.* **D 58** (1998) 096014 [[hep-ph/9802447](#)].
- [42] V. Cirigliano et al., *The $\langle SPP \rangle$ Green function and SU(3) breaking in K_{l3} decays*, *JHEP* **04** (2005) 006 [[hep-ph/0503108](#)].
- [43] H. Leutwyler and M. Roos, *Determination of the elements V_{Us} and V_{Ud} of the Kobayashi-Maskawa matrix*, *Z. Physik* **C 25** (1984) 91;
M. Jamin, J.A. Oller and A. Pich, *Order p^6 chiral couplings from the scalar $K\pi$ form factor*, *JHEP* **02** (2004) 047 [[hep-ph/0401080](#)];
D. Becirevic et al., *The $K \rightarrow \pi$ vector form factor at zero momentum transfer on the lattice*, *Nucl. Phys.* **B 705** (2005) 339 [[hep-ph/0403217](#)];
D.J. Antonio et al., *K_{l3} form factor with $N_f = 2 + 1$ dynamical domain wall fermions: a progress report*, [hep-lat/0702026](#).